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1.2 Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1 illustrates the process of mathematical modeling. Given a real-world problem, our first task is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from a library or the Internet or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

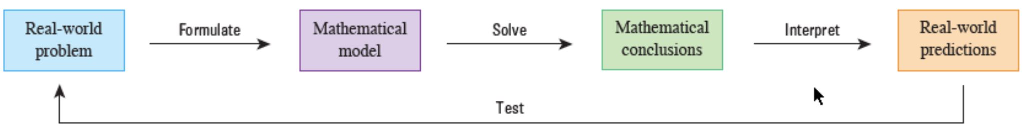


FIGURE 1 The modeling process

Linear Models

$f(x) = mx + b$

$\boxed{y - y_1 = m(x - x_1)}$

$Ax + By = C$

V EXAMPLE 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

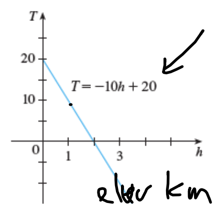
(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

$(0, 20)$

$(1, 10)$

T



elev km

V EXAMPLE 2 Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2008. Use the data in Table 1 to find a model for the carbon dioxide level.

TABLE 1

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1996	362.4
1982	341.2	1998	366.5
1984	344.4	2000	369.4
1986	347.2	2002	373.2
1988	351.5	2004	377.5
1990	354.2	2006	381.9
1992	356.3	2008	385.6
1994	358.6		

CO_2 level

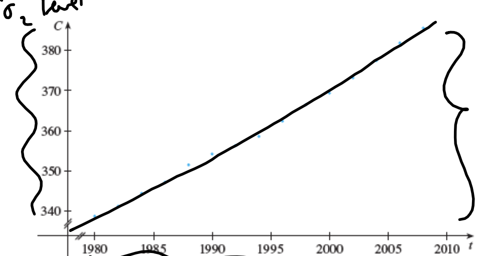
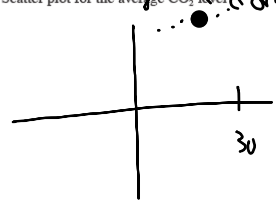


FIGURE 4 Scatter plot for the average CO_2 level

years domain

STAT EDIT

L1	L2	L3	L4



STAT CALC LinReg

y =

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Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

coefficients, degrees, quadratics, cubics

EXAMPLE 4 A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

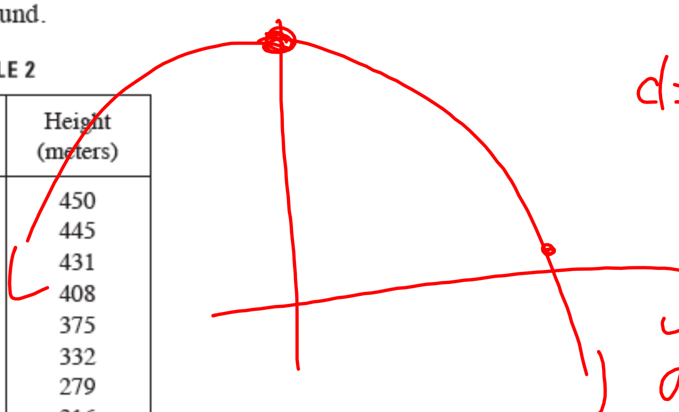


TABLE 2

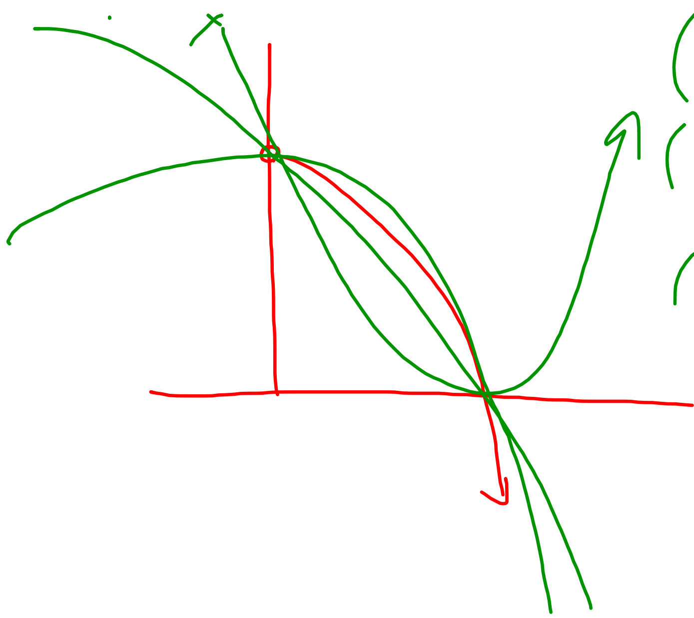
Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

$d = -\frac{1}{2} a t^2 + \cancel{} + h$

$d = -\frac{1}{2}(9.8) t^2 + 450$

$y = -4.9x^2 + 450$

$-4.9x^2 + .96x + 449.36$



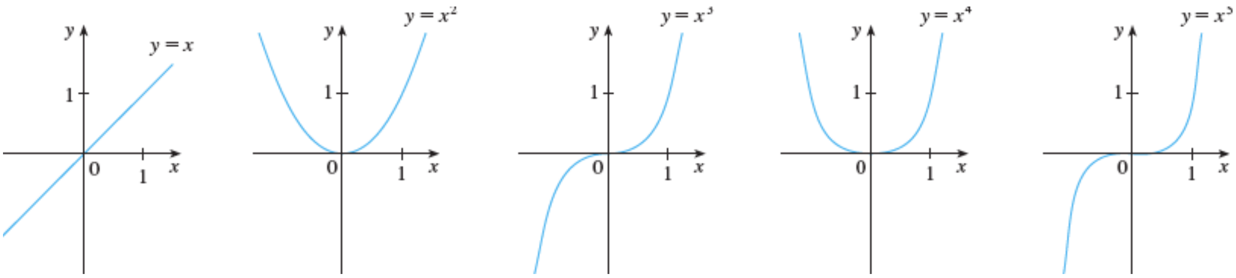
(1, 5)
(2, 7)
(9, 25)

Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

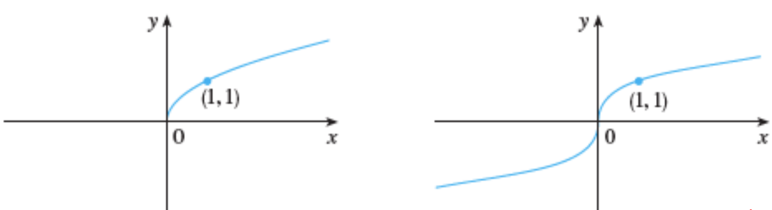
$f(x) = 5$ is a poly

(i) $a = n$, where n is a positive integer



$f(x) = x^a$

(ii) $a = 1/n$, where n is a positive integer

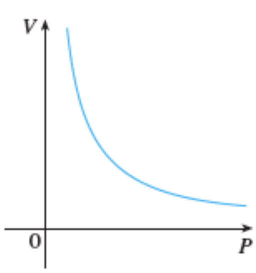


(a) $f(x) = \sqrt{x}$ $= x^{1/2}$

(b) $f(x) = \sqrt[3]{x}$ $x^{1/3}$

(iii) $a = -1$

$f(x) = \dfrac{1}{x}$ $x > 0$



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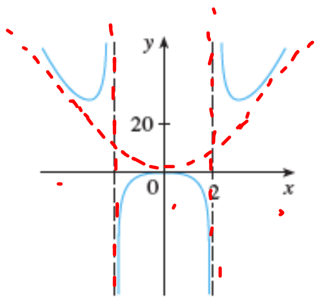
Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$



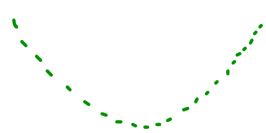
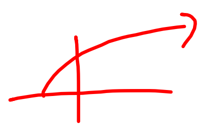


FIGURE 16

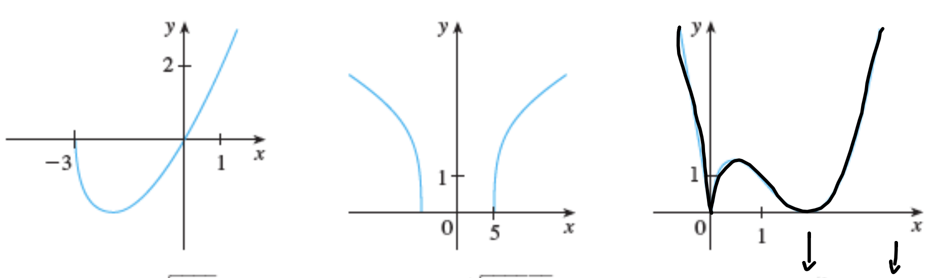
$f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$



Algebraic Functions

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$



(a) $f(x) = x\sqrt{x+3}$ (b) $g(x) = \sqrt[4]{x^2 - 25}$ (c) $h(x) = x^{2/3}(x-2)^2$

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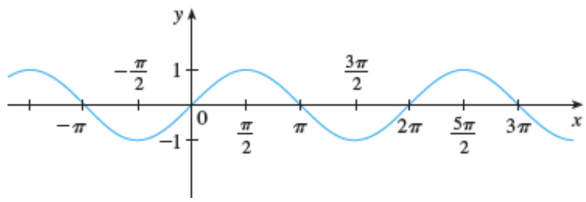
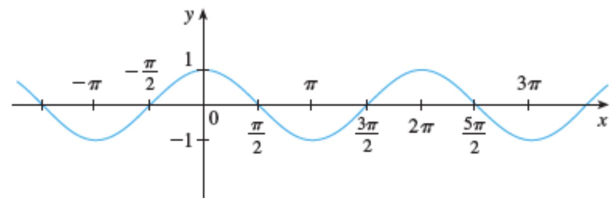
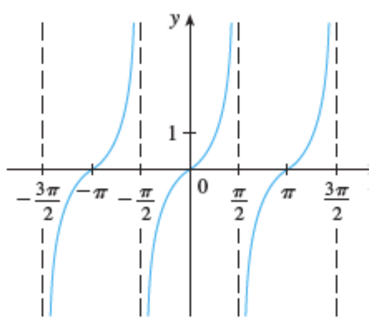
Trigonometric Functions





$$-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1$$

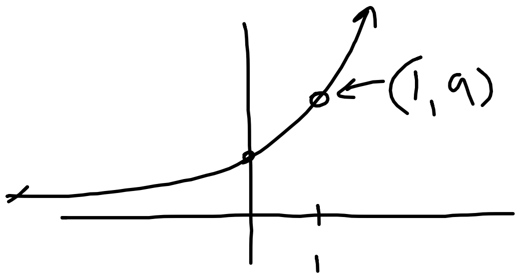
$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$



Exponential Functions

The **exponential functions** are the functions of the form $f(x) = a^x$.

$f(x) = a^x$

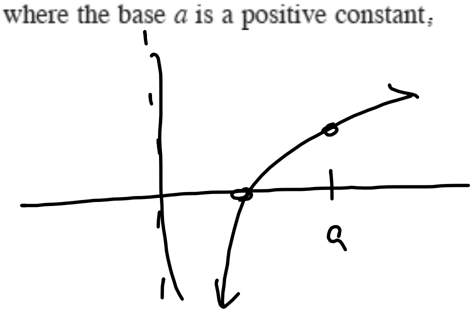


(0, 1)
(1, a)

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant,

$f(x) = \log_a x$



(1, 0)
(a, 1)

EXAMPLE 5 Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

1.2: 1, 3, 4, 5, 8, 11, 13, 15, 17, 19, 21, 27